Problem Set 3

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Problem 1 The Product of Graphs

When working in a new mathematical domain, we can use category theoretic concepts to help us understand common constructions.

For this problem, a graph $G = (G_v, G_e)$ consists of a set of vertices G_v and an incidence relation $G_e \subseteq G_v^2$ telling us when there is a (directed) edge between two vertices.

A graph homomorphism $\phi : G \to H$ is a function on the vertices $\phi : G_v \to H_v$ that preserves the incidence relation: if $(v, v') \in G_e$ then $(\phi(v), \phi(v')) \in H_e$.

There are many constructions that are called the "product" of graphs, but up to isomorphism only one that satisfies the universal mapping property of a product.

Consider the following two constructions.

For any two graphs G, H define the *box product* $G \,\square\, H$ to be the following graph:

1. The set of vertices is given by the cartesian product $(G \,\square\, H)_v = G_v \times H_v$.
2. There is an edge $((g, h), (g', h')) \in (G \,\square\, H)_e$ if and only if either of the following holds
   (a) $(g, g') \in G_e$ and $h = h'$.
   (b) $g = g'$ and $(h, h') \in H_e$.

For any two graphs G, H define the *tensor product* $G \otimes H$ to be the following graph:

1. The set of vertices is given by the cartesian product $(G \otimes H)_v = G_v \times H_v$.
2. There is an edge $((g, h), (g', h')) \in (G \otimes H)_e$ if and only if $(g, g') \in G_e$ and $(h, h') \in H_e$.

Both of these are useful ways to combine two graphs together, but only one of them is the true categorical product.

1. Show that one of these two products ($\square$, $\otimes$) gives the categorical product of graphs, i.e., always satisfies the universal mapping property of a product.
2. For the other, briefly explain why it fails to be a categorical product.

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Problem 2

When programming we often have an implicit *invariant* on our datatypes that is too complex to encode in our type system, but is crucial for justifying the correctness or safety of the code.

This can be modeled by a category of *subsets* where an object (X, P) is a pair of a set X and a subset $P \subseteq X$ of those values that satisfy the invariant. A morphism $f : (X, P) \to (Y, Q)$ is a function $f : X \to Y$ that preserves the invariant: if $p \in P$ then $f(p) \in Q$.

We can generalize this from subsets to "sub-objects" in any category as follows. Let $\mathbb{C}$ be a category. A sub-object of an object A is a monomorphism $m : P \rightarrowtail A$. A morphism of sub-objects $(f, \phi) : (m : P \rightarrowtail A) \to (n : Q \rightarrowtail B)$ consists of a morphism $f : A \to B$ and a morphism $\phi : P \to Q$ such that the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\ \phi\ } & Q \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle n} \\ A & \xrightarrow{\ f\ } & B \end{array}$$

The defines for any category $\mathbb{C}$ a category $\mathrm{Sub}(\mathbb{C})$ of subobjects. Composition and identity are defined the same as in the arrow category (Awodey Chapter 1.6).

For $\mathbb{C} = \mathrm{Set}$, $\mathrm{Sub}(\mathrm{Set})$ this generalizes our category of subsets slightly to allow arbitrary injective functions $m : P \rightarrowtail A$ as objects. We can view such a function as a presentation of the subset that is its image $\{x \in A | \exists p \in P. m(p) = x\}$. Then the property about ϕ demonstrates that f preserve this subset.

Let $\mathbb{C}$ be any category with an initial object 0, terminal object 1 and for any pair of objects $A, B \in \mathbb{C}$ a product $A \times B$.

- Prove that $\mathrm{Sub}(\mathbb{C})$ has an initial object, terminal object and all (binary) products.

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