Problem Set 6

Mar 20, 2022

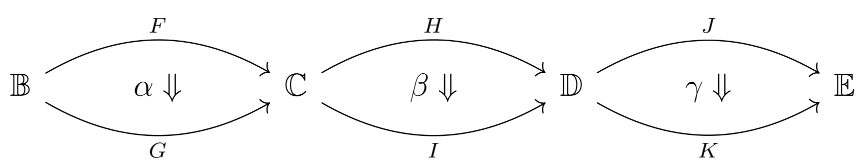
Homework is due the midnight before class on March 29.

Problem 1 2-dimensional Category Theory

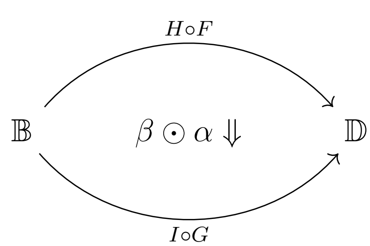
We showed in class that natural transformations form a category with "sequential composition". That is, if $\alpha : F \Rightarrow F'$ and $\alpha' : F' \Rightarrow F''$ then $\alpha' \circ \alpha : F \Rightarrow F''$ and this composition is associative and unital, forming the morphisms of the category of functors.

There is another way to compose natural transformations, called the *Godemont product* or the *parallel* composition.

For the following, let $\mathbb{B}, \mathbb{C}, \mathbb{D}, \mathbb{E}$ be categories, F, G, H, K, I, J be functors and natural transformations α, β, γ be natural transformations with domains and codomains given by the following diagram:



1. Show that for any such α, β we can define the *parallel composition* $\beta \odot \alpha : H \circ F \Rightarrow I \circ G$.



2. Prove that the parallel composition is associative: $\gamma \odot (\beta \odot \alpha) = (\gamma \odot \beta) \odot \alpha$

3. Prove that parallel composition is unital: $\alpha \odot \mathrm{id}_{\mathrm{id}_{\mathbb{B}}} = \alpha = \mathrm{id}_{\mathrm{id}_{\mathbb{B}}} \odot \alpha$

4. Prove the *interchange law* with the sequential composition holds:

$$(\beta' \circ \beta) \odot (\alpha' \circ \alpha) = (\beta' \odot \alpha') \circ (\beta \odot \alpha)$$

where $\beta' : I \Rightarrow I'$ and $\alpha' : G \Rightarrow G'$.

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Problem 2 Theorems for Free, Naturally

In a pure functional language without any reflection mechanisms, all definable polymorphic functions are natural. Phil Wadler, building on John Reynold's theory of parametricity, called this idea "theorems for free": just from the type of a polymorphic function, the naturality property[1] gives you properties that hold for every function of that type[Reynolds, 1983, Wadler, 1989].

For instance, given any function of the type $\forall X.X \to X$, i.e., a polymorphic function from X to X, generic in X, it can be shown that the function must be equivalent to the identity function. In effectful languages we need to weaken this statement. For instance in a language with recursive functions (and thus infinite loops) but no other effects, the only functions of type $\forall X.X \to X$ are the identity function and the function that always loops.

Your task is to prove these free theorems are true as consequences of naturality in different categories that model languages with effects. Note that we can model a polymorphic function $\forall X.X \to X$ as a natural transformation from $\mathrm{id}_{\mathbb{C}}$ to $\mathrm{id}_{\mathbb{C}}$ where $\mathbb{C}$ is the category modeling the functions in our programming language.

1. First, prove that the only natural transformation from $\mathrm{id}_{\mathrm{Set}}$ to $\mathrm{id}_{\mathrm{Set}}$ is the identity transformation.

2. Let Par be the category of sets and *partial* functions. Prove that the only natural transformations from $\mathrm{id}_{\mathrm{Par}}$ to $\mathrm{id}_{\mathrm{Par}}$ are the identity transformation and the transformation that is everywhere undefined.

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References

John C. Reynolds. Types, abstraction and parametric polymorphism. In *Information Processing 83, Proceedings of the IFIP 9th World Computer Congress, Paris, France*, 1983.

Philip Wadler. Theorems for free! In *Proceedings of the fourth international conference on Functional programming languages and computer architecture*, pages 347–359, 1989.

[1]more generally, dinaturality or more generally still, parametricity