Lecture 2: Models of Propositional Logic

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Last time, we talked about Intuitionistic Propositional Logic and ended with two questions:

1. Is IPL consistent? That is, can the judgment $\cdot \vdash \bot$ be derived)?

2. Does the law of the excluded middle follow from the rules for IPL? In other words, can we derive the judgment $\cdot \vdash X \vee \neg X$?

1 Introduction

1.1 Semantics

Semantics refers to the underlying mathematics associated with the syntax of a particular model. For example, IPL represents a model instance. As part of the syntax, we use the symbols $\{\wedge, \bot, \vee, \top, \supset\}$. However, these symbols are fundamentally meaningless without an underlying semantics.

1.2 Model Theory

It turns out that IPL presents a Heyting algebra. Heyting algebra are an abstract structure that can be used to provide underlying meaning for IPL propositions and implication. The study of these models is called Model Theory. The algebra is a way of formalizing the semantics.

A good analogy for Model Theory is linguistics. Even though English and Chinese aren't mutually intelligible, there are lots of similar ideas you can convey using sentences in both languages. Model Theory is akin to studying the underlying meaning of those sentences (semantics) and the particulars of how they are constructed in their respective languages (syntax).

IPL serves as a good introduction to more obscure instances of the Heyting algebra, because IPL is particularly well-known in the form of constructive logic. We will later find that some of the obscure instances will be useful for proving general properties about the Heyting algebra. This process will illustrate the usefulness of thinking of IPL in terms of Model Theory.

IPL support many models, including a Boolean model. This is because IPL is a subsystem of classical logic, where the law of the excluded middle (and, equivalently, double negation elimination) is not present. When we learned classical logic in EECS 203, we learned the definitions of implication, conjunction (logical AND), and disjunction (logical OR) in terms of their truth tables. In Category Theory, we take a more fundamental approach, defining these connectives in much more general terms.

2 A Boolean Semantics for IPL

In IPL, every closed proposition (i.e. a proposition with no unbounded variables) can be interpreted as either true or false. We can assign each proposition a value in $\{0, 1\}$, corresponding to false and true respectively. We use $[\![X]\!]$ to represent the *denotation* of a proposition X. We can now formalize our syntax for IPL as follows:

$$\begin{aligned} [\![\bot]\!] &:= 0 \\ [\![\top]\!] &:= 1 \\ [\![A \wedge B]\!] &:= \min([\![A]\!], [\![B]\!]) \\ [\![A \vee B]\!] &:= \max([\![A]\!], [\![B]\!]) \\ [\![A \supset B]\!] &:= \begin{cases} 1 & [\![A]\!] \leq [\![B]\!] \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

These definitions expand the evaluation of denotations to all judgments, including the connective forms.

We can extend this to an interpretation for a *context*, Γ. This is shorthand for the conjunction of many propositions.

$$\begin{aligned} \Gamma &= \{A_1, \ldots, A_n\} \text{ (may be empty)} \\ [\![\{A_1, \ldots, A_n\}]\!] &= \min\{[\![A]\!] \mid A \in \Gamma\} \end{aligned}$$

We will later define our semantics in terms of *meets* (greatest lower bounds) and *joins* (least upper bounds) respectively, which are a general property of Heyting algebras.

2.1 Proving Correctness

Theorem 1 (Soundness of Boolean Model)**.** *If $\Gamma \vdash A$ is provable in IPL, then $[\![\Gamma]\!] \leq [\![A]\!]$.*

To prove this we need to show that each rule of IPL gives us a true statement about booleans.

2.1.1 Assumption

We can show that the assumption rule is sound.

$$\frac{\cdot}{\Gamma, B \vdash B} \textsc{Assumption}$$

Recall that Γ is shorthand for the conjunction of a set of propositions. Under Boolean semantics, the above rule is interpreted as the statement:

$$\min\{\llbracket A \rrbracket \mid A \in \{\Gamma\} \cup B\} \leq \llbracket B \rrbracket$$

By definition of minimum, this holds.

2.1.2 Substitution

Some fundamental rules of inference, such as substitution, are *admissible*, meaning they can be derived from the existing rules of inference in the model. It is highly useful to minimize the number of rules, theorems, and assumptions tied to a particular model; in this case, we can reuse proofs of rules of inference to verify the correctness of admissible rules.

2.1.3 Top

In IPL, we had the rule

$$\frac{\cdot}{\Gamma \vdash \top}\ \top\text{-I}$$

For this to work in the Booleans, we need $\llbracket \Gamma \rrbracket \leq \llbracket \top \rrbracket$. This holds because $\llbracket \top \rrbracket = 1$, the maximum element in the set of Boolean values.

2.1.4 Conjunction

Recall the rule from IPL:

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge\text{-I}$$

In the Booleans, the following must hold:

$$\frac{\llbracket \Gamma \rrbracket \leq \llbracket A \rrbracket \qquad \llbracket \Gamma \rrbracket \leq \llbracket B \rrbracket}{\llbracket \Gamma \rrbracket \leq \min(\llbracket A \rrbracket, \llbracket B \rrbracket)}$$

This essentially follows from the definition of minimum; if $\llbracket \Gamma \rrbracket$ is smaller than $\llbracket A \rrbracket$ and $\llbracket B \rrbracket$, then it is smaller than the minimum between them. More formally, we know the minimum of a set is less than or equal to its constituents ($\min(\llbracket A \rrbracket, \llbracket B \rrbracket) \leq A, B$), so our proof follows from transitivity of $\leq$.

We saw before that the introduction rule for conjunction is invertible. Therefore, the validity of the conjunction-introduction rule implies the validity of the conjunction-elimination rule.

2.1.5 Disjunction

There are two ways to introduce disjunction in IPL:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee\text{-I-L}$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \vee\text{-I-R}$$

Let's examine the left introduction form.

$$\frac{[\![\Gamma]\!] \leq [\![A]\!]}{[\![\Gamma]\!] \leq \max([\![A]\!], [\![B]\!])}$$

This proof is slightly more complicated than the conjunction case, because $[\![\Gamma]\!]$ may be less than or equal to either $[\![A]\!]$ or $[\![B]\!]$. In both cases, though, it holds that $[\![\Gamma]\!]$ is less than the maximum between the two (by transitivity of $\leq$). Importantly, $[\![A]\!], [\![B]\!] \leq \max([\![A]\!], [\![B]\!])$. It suffices to use a proof-by-cases to show that this rule is valid.

2.2 Consistency of IPL

Theorem: Falsity cannot be derived from an empty context in IPL.

The Boolean semantics we have demonstrated for IPL are sufficient to prove consistency. By the soundness theorem of the boolean semantics, we can say the following:

$$\frac{\Gamma \vdash B}{[\![\Gamma]\!] \leq [\![B]\!]}$$

Now, substitute $\cdot$ for Γ and $\bot$ for B:

$$\frac{\cdot \vdash \bot}{[\![\cdot]\!] \leq [\![\bot]\!]}$$

By definition, $[\![\bot]\!] = 0$. So how can we interpret $[\![\cdot]\!]$? Under the standard interpretation that min represents the *meet* (or the infimum) of a set, we take the greatest lower bound of the empty set, which is 1, the greatest element of the booleans.

$$\frac{\cdot \vdash \bot}{1 \leq 0}$$

Therefore, because $1 \leq 0$, we have derived a contradiction, and so $\cdot \vdash \bot$ cannot be derivable.

2.3 Law of the Excluded Middle

Theorem: The law of the excluded middle is not admissible in IPL.

Recall that the LEM (law of the excluded middle) asserts $\Gamma \vdash A \vee \neg A$ for all propositions A in any context Γ. LEM is certainly *consistent* with IPL; in fact, $\neg\neg(X \vee \neg X)$ is a provable result in IPL. In other words, although $X \vee \neg X$ is not derivable, we can definitively say that it is *not false*. (This is also why double-negation elimination is tantamount to the LEM; if we accept the double-negation elimination rule, we could apply it and unlock the LEM for free.) Classical logic is a famous example of a Boolean algebra where LEM is perfectly consistent.

So what does it mean for the LEM to not be admissible in IPL? It means that it cannot not be derived from IPL. So to show this we need to come up with one particular proposition A and one context Γ and show that there is no proof of $\Gamma \vdash A \vee \neg A$. We will pick $\Gamma = \cdot$ and A to be a propositional variable X. So we seek to prove that $\cdot \vdash X$ is not provable in IPL. This may sound difficult to prove (how can you prove that something cannot be proven?), but with the help of Model Theory, we can do it. The idea is to construct a valid model of IPL, and then show that there is an element that gives a counterexample to the LEM (that is, that the LEM is invalid within that model), and interpret X as that element.

2.3.1 Generalization

Before we introduce a model that will let us prove the independence of LEM, we need to generalize a few ideas from our IPL semantics.

First, we need to extend our semantics to account for propositional variables. For this we require a function σ that maps propositions to values within a given model. Concretely, for IPL Boolean semantics, $\sigma : IPL \to \{0, 1\}$, and then we extend the denotations $[\![X]\!]_\sigma = \sigma(X)$.

We also provide a general formula for proving that a particular system is a valid instance of a model. We can use σ as an evaluation symbol, and we must also prove that all of the axioms of our model are valid within that system.

2.3.2 Sierpinski Model

The model we will use as a counterexample to LEM is the Sierpinski Model, which has *three* possible values: $\{0, 1/2, 1\}$ (where $0 \leq 1/2 \leq 1$). We can reuse much of the same syntax and semantics from IPL (therefore preserving correctness). Illustrative of the usefulness of generalization: we can freely modify the set of values (in this case, from $\{0, 1\}$) and maintain the fundamental characteristics of our model, because the rules and axioms within our model are proven independent of the set of values. Of course, in a more formal analysis, we would want to verify that all of our axioms remain valid.

The evaluation for implication must change to fit the Sierpinski model. Define a new operator $\Rightarrow$ which inherits most of its semantics from $\supset$ with a new evaluation:

$$[\![A \Rightarrow B]\!] := \begin{cases} 1 & [\![A]\!] \leq [\![B]\!] \\ [\![B]\!] & \text{otherwise} \end{cases}$$

This new operator is backwards-compatible, because the only time in Boolean algebra where the second case is taken is when $[\![B]\!] = 0$. When $[\![B]\!] = 1$, the first case always gets matched, because 1 is the maximum value in Boolean algebra.

Another important note is that $[\![\bot]\!] = 0$ and $[\![\top]\!] = 1$ still hold.

Recall the definition of the LEM:

$$\frac{\cdot}{\cdot \vdash X \vee \neg X}\ \text{LEM}$$

If we assign 1/2 to X, i.e., $\sigma(X) = 1/2$, we show that LEM cannot hold:

$$\frac{\cdot}{[\![\cdot]\!] \leq \max\{[\![X]\!]_\sigma, [\![X \Rightarrow \bot]\!]_\sigma\}}$$

From the definition of join, $[\![\cdot]\!] = 1$. Also, $[\![X]\!]_\sigma = \sigma(X) = 1/2$ and $[\![X \Rightarrow \bot]\!]_X = 0$ (since $\min(1/2, 0) = 0$; intuitively, X is not "false"). Therefore, this expression is equivalent to $1 \leq 1/2$, which concludes the proof by contradiction.

So if we can show that all of the rules of IPL can be interpreted in the Sierpinski model, then we can prove that LEM is not admissible.

3 Order Theory

In order to prove two important results for IPL, we first introduced the boolean model for IPL to show that IPL is consistent, then we used a contradiction in another instance of that model to show that LEM is independent of the model. How can we formalize the very idea of a model of IPL?

All of our models had conjunction, disjunction, and implication in common. Implication naturally lends itself to an idea of *ordering*, where certain propositions imply other propositions.

In the simplest case, in the Boolean model, we have 0 and 1 where $0 \leq 1$. However, we also have intermediate propositions that we can construct an order on. If $A \supset B$, then we can say $A \leq B$ (in this case, $\leq$ refers to order, not just evaluation, although the two ideas are quite connected). For the case of $\bot$ and $\top$, we can say that $\bot \leq \top$ $(0 \leq 1)$ but not $\top \leq \bot$ $(1 \leq 0)$.

Conjunction and disjunction also relate propositions in an ordered manner. For example, the conjunction $A \wedge B$ implies both A and B. Hence, $A \wedge B \leq A, B$. In the case of disjunction (which is dual to conjunction in a sense), we have that $A, B \leq A \vee B$.

3.1 Preorders and Posets

Definition 1. *A* preorder X *consists of*

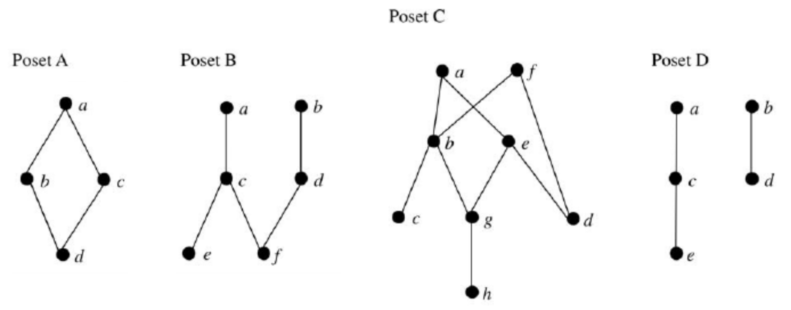
1. $|X|$*, a set*
2. $\leq_X$*, a binary relation on* $|X|$
3. *Reflexivity:* $x \leq_X x$ *for all* $x \in |X|$
4. *Transitivity:* $x \leq_X y \wedge y \leq_X z \Rightarrow x \leq_X z$ *for all* $x, y, z \in |X|$

Definition 2. *A* partial order*, or* poset *for short, is a preorder that additionally satisfies:*

5. *Antisymmetry:* $x \leq_X y \wedge y \leq_X x \Rightarrow x = y$ *for all* $x, y \in |X|$

3.2 Hasse Diagrams

A Hasse diagram visualizes a poset in terms of its ordering. Some examples:



3.3 Lindenbaum Algebra

Generally, we can take a logical system and turn it into a preorder. For example, in IPL, we have a set of propositions which are (typically) associated with each other using conjunctions, disjunctions, and implications. As explained previously, these connectives create a preorder over the propositions.

Our poset can be divided into equivalence classes (vertical layers) where each element in a given layer is greater than any element in the lower layers and lesser than any element in the higher layers. For IPL, the highest layer is the equivalence class $\{\top, \ldots\}$, all of the propositions equivalent to $\top$, and the lowest layer is $\{\bot, \ldots\}$ all of the propositions equivalent to $\bot$. This poset whose elements are equivalence classes of propositions and ordered by provable is called the *Lindenbaum algebra* of the logic.

3.4 Homomorphisms

A homomorphism is a mapping or transformation between two spaces that preserves their fundamental properties. In the case of preorders, we are interested in homomorphic functions which map each element without changing the overall ordering. We call these functions *monotone functions* (also known as order-preserving functions).

Definition 3. *If* P, Q *are preorders, a monotone function* $f : P \to Q$ *is a function of the underlying sets* $f : |P| \to |Q|$ *that satisfies the monotonicity property:* $\forall x, y \in |P|. x \leq_P y \Rightarrow f(x) \leq_Q f(y)$*.*

Homomorphisms are particularly useful for reasoning about infinite sets because they preserve certain abstract properties. Monotonicity is transitive; the composition of two monotone functions is also monotone.

A dual form of the monotone function is the *antitone* (or contravariant) function, which reverses the ordering. An example is negation over the integers. The composition of two antitone functions is monotone (since we reverse the order twice, restoring the original order).

3.5 Meets and Joins

A *meet* is a greatest lower bound on a set, and a *join* is a least upper bound. Formally, a meet on $S \subseteq X$ is a value x such that:

1. $x \leq y$ for all $y \in S$
2. $z \leq y \to z \leq x$ for all $y \in S, z \in X$

A join may be defined as follows:

1. $y \leq x$ for all $y \in S$
2. $y \leq z \to x \leq z$ for all $y \in S, z \in X$

Recall that earlier, in our Boolean model, we defined conjunction and disjunction as follows:

$$[\![A \wedge B]\!] := \min([\![A]\!], [\![B]\!])$$
$$[\![A \vee B]\!] := \max([\![A]\!], [\![B]\!])$$

These definitions can be generalized into meets and joins. In our generalization, we denote the conjunction of x, y to be the meet of $\{x, y\}$, and we denote the disjunction of x, y to be the join of $\{x, y\}$. This definition provides elegant evaluation semantics for Γ in IPL: we define $[\![\Gamma]\!]$ to be the meet of Γ.

A *meet semilattice* is a poset which has a meet for every subset. Dually, a *join semilattice* is a poset which has a join for every subset. A poset which is both a meet and join semilattice is called a *lattice*; such a poset has meets and joins on all subsets.

4 Heyting Algebras

The Heyting algebra is an even higher level of abstraction which generalizes our Boolean and Sierpinski models. This model will encapsulate all that we have discussed regarding Order Theory, meets, and joins. Recall that in our Boolean algebra for IPL, we defined conjunction and disjunction in terms of minimums and maximums.

With the help of Order Theory, we have found that IPL obey a more general idea of *ordering*, where propositions are ordered by logical connectives. Recall that we started this lecture asking whether IPL was consistent and whether LEM is independent of IPL semantics.

nWe have already generalized conjunction and disjunction semantics into meets and joins. The last step is to create a general model of what *implication* means.

A *Heyting implication* structure is a binary operation $x \to y$ satisfying $(z \wedge x) \leq y \equiv z \leq x \supset y$. This is the minimal necessary structure to allow for implication elimination (if P and $P \supset Q$, then Q), which is a fundamental axiom of Heyting algebras.

Definition 4. *A Heyting algebra is a poset with*

- *finite meets*
- *finite joins*
- *Heyting implications*

And we can show (next time) that Heyting algebras provide a canonical notion of model of IPL.