Lecture 3: Soundness, Completeness, Initiality of Heyting Algebra Semantics

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1 Partially Ordered Set

A *partial order* on a set X is a binary relation $\leq$ which is reflexive, transitive and anti-symmetric. A set X equipped with a partial order is called a *partially ordered set*, or somtimes a *poset*.

1.1 Meets, Joins, and Heyting Implication

Let $\mathbb{P}$ be a poset and $S \subseteq |\mathbb{P}|$, where $|\mathbb{P}| = \{*\}$, the underlying set of the poset.

Definition 1. *Z is a* meet *of S, if it's the greatest element of $\{x \in |\mathbb{P}|\, | x \leq S\}$. The meet of subset S is denoted as $\bigwedge S$.*

Definition 2. *Z is a* join *of S, if it's the greatest element of $\{x \in |\mathbb{P}|\, | x \leq S\}$. The join of subset S is denoted as $\bigvee S$.*

Definition 3. *Z is a* Heyting Implication *of $x, y \in |\mathbb{P}|$, if it's the greatest element of $\{w \in |\mathbb{P}|\, | w \wedge x \leq y\}$. The Heyting Implication of elements x, y is denoted as $x \Rightarrow y$.*

Suppose the subset S contains finite elements $x_1, x_2, \ldots, x_n$. The set operators meet and join can be correspondingly related to the binary operators conjunction and disjunction as shown in the following table.

Set Operator	Binary Operator	Transformation	Base Condition
meet $\bigwedge$	conjunction $\wedge$	$\bigwedge S = x_1 \wedge x_2 \wedge \ldots \wedge x_n$	$\bigwedge \emptyset = \top$
join $\bigvee$	disjunction $\vee$	$\bigvee S = x_1 \vee x_2 \vee \ldots \vee x_n$	$\bigvee \emptyset = \bot$

Interestingly, there is a mutual conversion between meets and joins in specic conditions: $\bigwedge_{\mathbb{P}} \varnothing = \bigvee \mathbb{P}$ and $\bigvee_{\mathbb{P}} \varnothing = \bigwedge \mathbb{P}$

1.2 Lattice

Definition 4. Lattice *is a poset with finite meets and finite joins.*

Definition 5. Complete Lattice *is a poset with all meets and all joins.*

Definition 6. $\wedge$-Semilattice *is a poset with finite meets.*

Definition 7. $\vee$-Semilattice *is a poset with finite joins.*

Definition 8. Heyting Algebra *is a lattice with all Heyting Implications.*

Notice that *finite* means the property exists in any finite subset S, while *all* means the property exists in any subset S, no matter finite or infinite.

2 Heyting Algebra Semantics

2.1 Soundness Theorem

Given a set Σ_0 of propositional variables, define IPL_{Σ_0} to be the propositions of IPL with variables drawn from Σ_0. An axiom relative to Σ_0, is an element of $\mathrm{IPL}^*_{\Sigma_0} \times \mathrm{IPL}_{\Sigma_0}$ which we write as $A_1, \ldots \Rightarrow A'$. Then an IPL signature is a pair of a set Σ_0 and a set of axioms Σ_1 relative to Σ_0.

Theorem 1 (Interpretation of Propositions in HA, Soundness)**.** *Let Σ be an IPL signature.*

Given any Heyting algebra $\mathbb{P}$ and a function $\sigma : \Sigma_0 \to |\mathbb{P}|$ that interprets the variables, we can extend this to a function $[\![\cdot]\!]_\sigma : IPL_{\Sigma_0} \to |\mathbb{P}|$ as follows:

$$
\begin{aligned}
[\![X]\!]_\sigma &= \sigma(X) \\
[\![\top]\!]_\sigma &= \top_{\mathbb{P}} \\
[\![A \wedge B]\!]_\sigma &= [\![A]\!]_\sigma \wedge_{\mathbb{P}} [\![A]\!]_\sigma \\
[\![\bot]\!]_\sigma &= \bot_{\mathbb{P}} \\
[\![A \vee B]\!]_\sigma &= [\![A]\!]_\sigma \vee_{\mathbb{P}} [\![A]\!]_\sigma \\
[\![A \supset B]\!]_\sigma &= [\![A]\!]_\sigma \Rightarrow_{\mathbb{P}} [\![A]\!]_\sigma
\end{aligned}
$$

This interpretation is extended to contexts by $[\![\Gamma]\!]_{\mathbb{P}} = \bigwedge_{\mathbb{P}}\{[\![A]\!]_\sigma | A \in \Gamma\}$

If for each axiom $\Gamma \Rightarrow A \in \Sigma_1$ it is the case that $[\![\Gamma]\!]_\sigma \leq_{\mathbb{P}} [\![A]\!]_\sigma$, then if $\Gamma \vdash A$ is provable in IPL, $[\![\Gamma]\!]_\sigma \leq_{\mathbb{P}} [\![A]\!]_\sigma$.

Proof. Given $\Gamma \vdash A$, we seek to prove that $[\![\Gamma]\!]_\sigma \leq_{\mathbb{P}} [\![A]\!]_\sigma$. Since proofs of $\Gamma \vdash A$ are inductively defined we can show this by *structural induction* over such proofs. At a high level this means that for each rule

$$\frac{\Gamma_1 \vdash A_1 \qquad \Gamma_2 \vdash A_2 \qquad \ldots \qquad \Gamma_n \vdash A_n}{\Gamma \vdash A}$$

we show the implication

$$\frac{[\![\Gamma_1]\!]_\sigma \leq_{\mathbb{P}} [\![A_1]\!]_\sigma \qquad [\![\Gamma_2]\!]_\sigma \leq_{\mathbb{P}} [\![A_2]\!]_\sigma \qquad \ldots [\![\Gamma_n]\!]_\sigma \leq_{\mathbb{P}} [\![A_n]\!]_\sigma}{[\![\Gamma]\!]_\sigma \leq_{\mathbb{P}} [\![A]\!]_\sigma}$$

That is, assuming all of the inductive hypotheses above the line are true then we can prove the conclusion below the line.

We give some example cases □

We call the inputs to the soundness theorem: a Heyting algebra $\mathbb{P}$ with interpretation σ of Σ_0 such that all of the axioms in Σ_1 are satisfied in $\mathbb{P}$ a *model* of the logic IPL generated by Σ. Then the soundness theorem says that this notion of model is sound: any theorem that is provable in the logic is true in every model.

2.1.1 Applications of the Soundness Theorem

Theorem 2. *The $\bot$-Introduction Law $\cdot \vdash \bot$ is not provable in pure IPL (IPL with the empty signature $(\emptyset, \emptyset)$).*

Proof. Assume to the contrary that $\cdot \vdash \bot$ is provable.

Define the *boolean poset* $\mathbb{B}$ to be the set $\{0, 1\}$ with the order inherited from the integers (i.e., $0 \leq 1$). $\mathbb{B}$ is a Heyting algebra (proof left as exercise).

By the soundness theorem, since $\cdot \vdash \bot$ is provable, $\bigwedge_{\mathbb{B}} \emptyset \leq_{\mathbb{B}} \bot_{\mathbb{B}}$ must be true. But $\bigwedge \emptyset = \top_{\mathbb{B}} = 1$ and $\bot_{\mathbb{B}} = 0$, so this would mean that $1 \leq_{\mathbb{B}} 0$ which is false, so we have derived a contradiction. □

Theorem 3. *The Law of Excluded Middle $\cdot \vdash X \vee (X \supset \bot)$ is not provable in IPL with one propositional variable X and no axioms.*

Proof. Assume to the contrary that $\cdot \vdash X \vee (X \supset \bot)$ is provable.

Define the *Sierpinski poset* $\mathbb{S}$ to be the set $\{0, \frac{1}{2}, 1\}$ with the order inherited from the rationals (i.e., $0 \leq \frac{1}{2} \leq 2$). Define an interpretation σ by $\sigma(X) = \frac{1}{2}$. The Sierpinski poset is a Heyting algebra (exercise).

Then by the soundness theorem, since $\cdot \vdash X \vee (X \supset \bot)$ is provable it must be the case that $1 \leq_{\mathbb{S}} \frac{1}{2} \vee_{\mathbb{S}} (\frac{1}{2} \Rightarrow_{\mathbb{S}} 0)$. However $\frac{1}{2} \Rightarrow_{\mathbb{S}} 0 = 0$ and so $\frac{1}{2} \vee (\frac{1}{2} \Rightarrow 0) = \frac{1}{2}$. This implies that $1 \leq_{\mathbb{S}} \frac{1}{2}$ which is again a contradiction. □

2.2 Completeness Theorem

The soundness theorem says provable statements are true in every model. The completeness theorem gives us a converse: if a statement is true in every model then it must be provable.

Theorem 4. *Let Σ be an IPL signature and Γ, A well-formed context and proposition in IPL generated from Σ. Then if for every model $(\mathbb{P}, \sigma)$*

$$[\![\Gamma]\!]_\sigma \leq_{\mathbb{P}} [\![A]\!]_\sigma$$

is true, $\Gamma \vdash A$ is provable in IPL generated by Σ.

Proof. We show this by defining a single model, the *Lindenbaum algebra* $\mathcal{L}$ where truth in $\mathcal{L}$ implies provability in IPL. First, define a preorder whose elements are simply the propositions of IPL generated by Σ and where $A \leq_{\mathcal{L}} B$ is defined to be provability: $A \vdash B$ is provable. Then using the substitution and assumption rules this is a preorder:

- **Reflexive.** For any proposition A,

$$\frac{}{A \vdash A}\ \textsc{Assumption}$$

- **Transitive.** For any proposition A, B, C,

$$\frac{A \vdash C \qquad \dfrac{B \vdash C}{A, B \vdash C}\ \textsc{Weakening}}{A \vdash C}\ \textsc{Substitution}$$

  Then according to our definition of $\leq_{\mathcal{L}(\text{IPL})}$, $\forall A, B, C, A \leq_{\mathcal{L}(\text{IPL})} B, B \leq_{\mathcal{L}(\text{IPL})} C : A \leq_{\mathcal{L}(\text{IPL})} C$.

However, this is not a poset since there are propositions such as $\top$ and $\top \wedge \top$ that are equi-provable but not equal. Then define $\mathcal{L}$ to be the corresponding poset quotiented by the equivalence $A \cong B$ if $A \vdash B$ and $B \vdash A$.

Then we need to show that $\mathcal{L}$ is a Heyting algebra. To do this we show $A \wedge B$ is the binary meet in $\mathcal{L}$, $A \vee B$ is the binary join, $\top$ is the empty meet, $\bot$ is the empty join and $A \supset B$ is the Heyting implication. We show one example, the binary meet.

We need to show that $A \wedge B$ is the greatest proposition such that that $A \wedge B \vdash A$, and $A \wedge B \vdash B$.

1. First, $A \wedge B \vdash A$ using the elimination rule and assumption:

$$\frac{\dfrac{}{A \wedge B \vdash A \wedge B}\ \textsc{Assumption}}{A \wedge B \vdash A}\ \wedge\text{E1}$$

   $A \wedge B \vdash B$ follows similarly.

2. Assume that $C \vdash A$ and $C \vdash B$. Then we need to show that $C \vdash A \wedge B$. But this is precisely the $\wedge I$ rule.

Finally, we can define the interpretation of variables $\sigma(X) = X$ as simply themeselves, and therefore by induction we can see that $[\![A]\!]_\sigma = A$. Additionally, all axioms are clearly satisfied using the corresponding axiom rule.

Finally, to complete the proof, assume $[\![\Gamma]\!]_\sigma \leq_{\mathcal{L}} [\![A]\!]_\sigma$. Then $\bigwedge\{B | B \in \Gamma\} \vdash A$, i.e. if $\Gamma = B_0, \ldots$ then we have

$$B_0 \wedge (\ldots \top) \vdash A$$

Then the final step is to show that the rule

$$\frac{B_0, \ldots \vdash A}{B_0 \wedge (\ldots \top) \vdash A} \bigwedge\mathrm{L}$$

is admissible by induction on Γ. □

2.3 Initiality/Freeness Theorem

We conclude by noting the following "categorical" re-formulation of the soundness and completeness theorems.

Definition 9. *Let $\mathbb{P}$ and $\mathbb{Q}$ be Heyting algebras. A* Heyting algebra homomorphism *is a monotone function $\varphi : \mathbb{P} \to \mathbb{Q}$ that furthermore preserves finite meets, finite joins and Heyting implications. I.e.,*

- $\varphi(\top_{\mathbb{P}}) = \top_{\mathbb{Q}}$.
- $\varphi(\bot_{\mathbb{P}}) = \bot_{\mathbb{Q}}$.
- $\varphi(A \wedge_{\mathbb{P}} B) = \varphi(A) \wedge_{\mathbb{Q}} \varphi(B)$.
- $\varphi(A \vee_{\mathbb{P}} B) = \varphi(A) \vee_{\mathbb{Q}} \varphi(B)$.
- $\varphi(A \Rightarrow_{\mathbb{P}} B) = \varphi(A) \Rightarrow_{\mathbb{Q}} \varphi(B)$.

If $(\mathbb{P}, \sigma)$ and $(\mathbb{Q}, \tau)$ are models of Σ, then a homomorphism of models *is a Heyting algebra homomorphism $\phi : \mathbb{P} \to \mathbb{Q}$ that furthermore satisfies $\phi(\sigma(X)) = \tau(X)$ for every $X \in \Sigma_0$.*

Theorem 5. *Fix a signature Σ, and let $(\mathcal{L}, \iota)$ be the Lindenbaum algebra, where ι is the subset inclusion of propositional variables into propositions. For any model $(\mathbb{P}, \sigma)$ of Σ there is a* unique *homomorphism of models $[\![\cdot]\!]_{(\mathbb{P},\sigma)} : (\mathcal{L}, \iota) \to (\mathbb{P}, \sigma)$ that satisfies $[\![\iota(X)]\!]_{(\mathbb{P},\sigma)} = \sigma X$.*

We can visualize this as the following diagram. We say that $\overline{\sigma}$ is the unique homomorphism making the following diagram "commute":

$$\begin{tikzcd} \mathcal{L} \arrow[rr, "\overline{\sigma}"] & & \mathbb{P} \\ & \Sigma_0 \arrow[ul, "\iota"] \arrow[ur, "{[\![\cdot]\!]}"'] & \end{tikzcd}$$

Proof. $[\![]\!]_{(\mathbb{P},\sigma)}$ is the interpretation function defined in the statement of the soundness theorem. It clearly preserves the Heyting algebra and satisfies $[\![X]\!]_{(\mathbb{P},\sigma)} = \sigma X$ by definiiton.

Furthermore, to prove $[\![\cdot]\!]_{(\mathbb{P},\sigma)}$ is unique, we assume there is some other homomorphism φ that extends σ and we must show $\varphi = [\![\cdot]\!]_{(\mathbb{P},\sigma)}$. This follows by induction on A:

1. $\varphi(X) = \sigma(X) = [\![\iota(X)]\!]_{(\mathbb{P},\sigma)}$ by assumption.

2. $\varphi(\top) = \top_{\mathbb{P}} = [\![\top]\!]_{(\mathbb{P},\sigma)}$ since $\varphi, [\![\cdot]\!]_{(\mathbb{P},\sigma)}$ are Heyting algebra homomorphisms.

3. $\varphi(A \wedge B) = \varphi(A) \wedge_{\mathbb{P}} \varphi(B)$ and $[\![A \wedge B]\!] = [\![A]\!] \wedge_{\mathbb{P}} [\![B]\!]$ since $\varphi, [\![\cdot]\!]$ are Heyting algebra homomorphisms. Then the result follows because $\varphi A = [\![A]\!]$ and $\varphi B = [\![B]\!]$ by inductive hypothesis.

4. The remaining cases are similar to the previous.

□

3 Syntax and Semantics

This situation where we have an initiality theorem for a logic with a certain notion of model will be a running theme throughout the course. We will study syntactic systems alongside a notion of semantic model and prove such an initiality theorem. This situation fruitfully benefits both sides: to a mathematician it says we can use the syntax to more easily make constructions in our semantic models, and to a computer scientist it says we can use semantic tools to more easily prove facts about our syntactic systems.

Already, we can see some more examples of this situation by taking subsystems of IPL and considering models, each of which will be posets with structure already present in a Heyting algebra. The following table shows some subsystems of Intuitionistic Propositional Logic and their corresponding order-theoretic semantics for which a similar initiality theorem can be proven.

Syntax / Logic	Semantics / Models
IPL () or IPL $(\wedge, \top)$	$\wedge$-Semilattice
IPL $(\wedge, \top, \supset)$	$\wedge$-Semilattice with Heyting Implication
IPL $(\wedge, \vee, \top, \bot)$	Distributive Lattice
IPL $(\wedge, \vee, \top, \bot, \supset)$	Heyting Algebra

The reason that the system IPL() with no connectives works is the same as IPL$(\top, \wedge)$ is that IPL() already implicitly contains something like $\top$ and $\wedge$ in the form of the empty context and concatenation of contexts.

To get a system whose semantics are all posets rather than only $\wedge$-Semilattices, we could define a very weak where all judgments are of the form $A \vdash B$, i.e., where we always have exactly one assumption. Such a system could be called *Unary* propositional logic (UPL).

Syntax / Logic	Semantics / Models
UPL ()	Posets
UPL $(\wedge, \top)$	$\wedge$-Semilattices
UPL $(\vee, \bot)$	$\vee$-Semilattices
UPL $(\wedge, \vee, \top, \bot)$	Lattices (not necessarily distributive)

Open-ended exercise: how would you design the syntax of UPL so that these initiality theorems hold? Why does IPL $(\wedge, \vee, \top, \bot)$ have distributive lattices as models but UPL $(\wedge, \vee, \top, \bot)$ has lattices that might not be distributive?