Problem Set 3

Released: February 3, 2023
Due: February 13, 2023, 11:59pm

Submit your solutions to this homework on Canvas in a group of 2 or 3. Your solutions must be submitted in pdf produced using LaTeX.

Problem 1 Sections, Retraction, Isomorphisms

Riehl Category Theory in Context, Exercise 1.1.i (page 8).

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Problem 2 Representable Functors

Riehl Category Theory in Context, Exercise 1.3.iv (page 22).

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Problem 3 Initiality of Pure Unary Type Theory

For this exercise you will work thorugh the full details for a very simple initiality proof for a type theory: pure unary type theory. Unary because every term will have exactly one free variable, and pure meaning no connectives.

A non-equational *pure unary type theory signature*[1] $\Sigma = (\Sigma_0, \Sigma_1)$ consists of

1. A set Σ_0 of base types
2. A family $\Sigma_1 : \Sigma_0^2 \to \text{Set}$ of *function symbols*, i.e., for each $X, Y \in \Sigma_0$ a set $\Sigma_1(X, Y)$ of function symbols with domain X and codomain Y.

Unary type theory relative to $\Sigma = (\Sigma_0, \Sigma_1)$ is a simplified version of simple type theory defined as follows. The only types A, B, C are base types, i.e., elements of Σ_0. The terms are freely generated by two constructions:

[1] note that this is the same data as a directed multigraph

$$\frac{}{x : A \vdash x : A}\ \textsc{Var} \qquad\qquad \frac{x : A \vdash M : B \qquad f \in \Sigma_1(B, C)}{x : A \vdash f(M) : C}\ \textsc{FunSym}$$

This defines for each x, A, B a set $\mathrm{Term}(x, A, B)$ of terms $x : A \vdash M : B$. So the terms of this language are simply a sequence of function symbols applied to a variable, for instance $f(g(h(x)))$.

Given $x : A \vdash M : B$ and $y : B \vdash N : C$ we can define by structural recursion on N, the substitution $N[M/x]$ as follows:

$$f(N')[M/x] = f(N'[M/x])$$
$$x[M/x] = M$$

No equational theory is needed for unary type theory if we only use a non-equatinal signature. An equational unary type theory signature $(\Sigma_0, \Sigma_1, \Sigma_2)$ extends a non-equational signature (Σ_0, Σ_1) with a collection of *axioms*, i.e., for each x, A, B a subset $\Sigma_2(x, A, B) \subseteq \mathrm{Term}(x, A, B) \times \mathrm{Term}(x, A, B)$. If $M, N \in \Sigma_2(x, A, B)$ we think of this as an axiom $x : A \vdash M = N : B$.

The equational theory of unary type theory generated by an equational signature $(\Sigma_0, \Sigma_1, \Sigma_2)$ is generated by the following rules:

$$\frac{x : A \vdash M : B}{x : A \vdash M = M : B}\ \textsc{Reflexivity} \qquad\qquad \frac{x : A \vdash M = N : B}{x : A \vdash N = M : B}\ \textsc{Symmetry}$$

$$\frac{x : A \vdash M = N : B \qquad x : A \vdash N = P : B}{x : A \vdash M = P : B}\ \textsc{Transitivity}$$

$$\frac{x : A \vdash M = M' : B \qquad f \in \Sigma_1(B, C)}{x : A \vdash f(M) = f(M') : C}\ \textsc{Cong}$$

$$\frac{(M, M') \in \Sigma_2(y, B, C) \qquad x : A \vdash N = N' : B}{x : A \vdash M[N/y] = M'[N'/y] : C}\ \textsc{Axiom}$$

An interpretation of a non-equational signature Σ in a category $\mathcal{C}$, written $i : \Sigma \rightsquigarrow \mathcal{C}$ consists of

1. An interpretation of base types $i_0 : \Sigma_0 \to \mathcal{C}_0$
2. An interpretation of function symbols, i.e., for each $A, B \in \Sigma_0$ $i_1^{A,B} : \Sigma_1(A, B) \to \mathcal{C}_1(i(A), i(B))$.

Note that if $F : \mathcal{C} \to \mathcal{D}$ is a functor, and $i : \Sigma \rightsquigarrow \mathcal{C}$ is an interpretation, we can compose these to get an interpretation $F \circ i : \Sigma \rightsquigarrow \mathcal{D}$:

1. $(F \circ i)_0(A) = F_0(i_0(A))$

2. $(F \circ i)_1^{A,B}(f) = F_1^{i_0(A),i_0(B)}(i_1^{A,B}(f))$

Your goal is to show that unary type theory generates the *free category* from a signature Σ. We proceed in two steps: first a theorem for non-equational signatures, and then extending this to equational signatures.

Let $\Sigma = (\Sigma_0, \Sigma_1)$ be a non-equational signature.

We define a category $\mathcal{L}(\Sigma)$ whose objects are the base types $A \in \Sigma_0$ and whose morphisms $M : \mathcal{L}(\Sigma)(A, B)$ are terms $x : A \vdash M : B$ for a globally fixed variable x. Identity is the variable itself $x : A \vdash x : A$ and composition is substitution $M[N/x]$. This definition is unital and associative by a simplified argument to the previous homework.

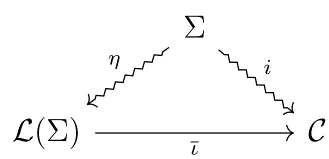
Part 1:

- Define an interpretation $\eta : \Sigma \rightsquigarrow \mathcal{L}(\Sigma)$ that is *universal* in the following sense: for any interpretation $i : \Sigma \rightsquigarrow \mathcal{C}$, there is a *unique* functor $\bar{\imath} : \mathcal{L}(\Sigma) \to \mathcal{C}$ such that $\bar{\imath} \circ \eta = i$.

  This consists of three parts:

  1. Define η
  2. Construct for any i, an $\bar{\imath}$ that satisfies $\bar{\imath} \circ \eta = i$
  3. Show that $\bar{i}$ is unique, i.e., for any i, and $F : \mathcal{L}(\Sigma) \to \mathcal{C}$ satisfying $F \circ \eta = i$, we can prove $F = \bar{i}$.

  Diagrammatically, $\bar{\imath}$ is the unique functor that makes the following diagram commute:



With this established, let $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2)$ now be an *equational* signature. An interpretation of Σ in a category $\mathcal{C}$, written again as $i : \Sigma \rightsquigarrow \mathcal{C}$ is an interpretation of the non-equational signature $(i_0, i_1) : (\Sigma_0, \Sigma_1) \rightsquigarrow \mathcal{C}$ that additionally satisfies:

- For every $(M, N) \in \Sigma_2(x, A, B)$, $\overline{(i_0, i_1)}(M) = \overline{(i_0, i_1)}(N)$.

We can then define a category $\widetilde{\mathcal{L}}(\Sigma)$ as follows:

1. The objects are the base types $A \in \Sigma_0$.
2. The morphisms $[M] : \widetilde{\mathcal{L}}(\Sigma)(A, B)$ are *equivalence classes* of terms $x : A \vdash M : B$ modulo the equivalence relation induced by the equational theory $x : A \vdash M = N : B$.

   Identity is given by the equivalence class of a variable, and composition is given by composition. To prove this is valid we need to show that composition respects the equational theory, which is precisely the same as showing that the following rule is admissible:

$$\frac{y : B \vdash M = M' : C \qquad x : A \vdash N = N' : B}{x : A \vdash M[N/y] = M'[N'/y]} \textsc{SubstCong}$$

BONUS: For two bonus points, show that SubstCong is admissible in Unary type theory (by induction on the proof that $M = M'$)

Part 2:

- Prove that the interpretation $\eta : (\Sigma_0, \Sigma_1) \rightsquigarrow \mathcal{L}(\Sigma_0, \Sigma_1)$ you defined above extends to an interpretation $\tilde{\eta} : (\Sigma_0, \Sigma_1, \Sigma_2) \rightsquigarrow \widetilde{\mathcal{L}}(\Sigma_0, \Sigma_1, \Sigma_2)$, i.e., verify that the interpretation η additionally satisfies the axioms in Σ_2.
- Show that for any interpretation $i : (\Sigma_0, \Sigma_1, \Sigma_2) \rightsquigarrow \mathcal{C}$, the above definition of the functor $\bar{\imath} : \mathcal{L}(\Sigma_0, \Sigma_1) \to \mathcal{C}$ restricts to a functor $\tilde{\imath} : \widetilde{\mathcal{L}}(\Sigma_0, \Sigma_1, \Sigma_2) \to \mathcal{C}$. That is, show that your definition of $\bar{\imath}$ respects the equational theory: if $x : A \vdash M = N : B$ then $\bar{\imath}(M) = \bar{\imath}(N)$.

Then the fact that $\tilde{\imath}$ is the unique functor satisfying $\tilde{\imath} \circ \tilde{\eta} = \iota$ follows by a similar argument to the case of $\bar{\imath}$.

Unary type theory without axioms presents a free category, analogous to other free algebraic structures such as free groups. Unary type theory *with* axioms is similarly analogous to algebraic structures presented by generators and relations.

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